Products of Radon measures (Ch. 7.4)

Recall. If X, Y are spaces w/ σ-algebras $\mathcal{M}, \mathcal{N}$ resp., and μ, ν are measures on $(X, \mathcal{M})$, $(Y, \mathcal{N})$, then the product measure $\mu \times \nu$ is a measure $\mu \times \nu$ on $X \times Y$ w/ σ-alg. $\mathcal{M} \otimes \mathcal{N}$ s.t. $(\mu \times \nu)(A \times B) = \mu(A)\nu(B)$; moreover, if μ, ν are σ-finite, then $\mu \times \nu$ is unique. Here, $\mathcal{M} \otimes \mathcal{N}$ is the σ-alg. generated by the sets $\{A \times Y \;\&\; X \times B : A \in \mathcal{M}, B \in \mathcal{N}\} = \{\pi_X^{-1}(A), \pi_Y^{-1}(B) : A \in \mathcal{M}, B \in \mathcal{N}\}$.

$$\begin{tikzcd} & X \times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X & & Y \end{tikzcd}$$

I. Construction of the Radon product.

Let μ, ν be Radon measures on X, Y, resp. The product $\mu \times \nu$ is a measure on $\mathcal{B}_X \otimes \mathcal{B}_Y$ in $X \times Y$.

Thm 1. (i) $\mathcal{B}_X \otimes \mathcal{B}_Y \subseteq \mathcal{B}_{X \times Y}$.

(ii) $\mathcal{B}_X \otimes \mathcal{B}_Y = \mathcal{B}_{X \times Y}$ if X, Y are 2^{nd} countable.

(iii) If X, Y are 2^{nd} countable, μ, ν are Radon measures on X, Y, then $\mu \times \nu$ is Radon on $X \times Y$.

Pf. Pfs of (i), (ii) follows from arguments similar to those in Ch. 7 of Folland. (iii) follows easily from Thm 7.8. $\blacksquare$

Rem. In general (if X, Y are not 2^{nd} countable), $\mathcal{B}_X \otimes \mathcal{B}_Y \neq \mathcal{B}_{X\times Y}$ and $\mu \times \nu$ is not Borel (and hence not Radon).

To construct Radon meas. on $X\times Y$ out of Radon meas. μ, ν on X, Y, we proceed as follows.

$(\varphi\psi)(x,y) = \varphi(x)\psi(y)$

Prop 1. The vector space $\mathcal{P} \subseteq C_c(X\times Y)$ spanned by products $\varphi\psi$, where $\varphi \in C_c(X)$, $\psi \in C_c(Y)$, is dense in $C_c(X\times Y)$. Moreover, for any $f \in C_c(X\times Y)$ and open $U \supseteq \pi_X(\mathrm{supp} f)$, $V \supseteq \pi_Y(\mathrm{supp} f)$ one may approximate f by $F \in \mathcal{P}$ w/ $\mathrm{supp}\, F \subseteq U\times V$.

Pf. Pick $f \in C_c(X\times Y)$, $\varepsilon > 0$, and U, V as above. Since X, Y are LCH, WLOG assume $\overline{U}, \overline{V}$ cpct. $\Rightarrow$ $\overline{U}\times\overline{V}$ cpct.

Let $\mathcal{P}_{\bar{U}\times\bar{V}}$ be span of gh, $g\in\mathcal{C}(\bar{U})$, $h\in\mathcal{C}(\bar{V})$. Clearly $\mathcal{P}_{\bar{U}\times\bar{V}}$ is subalgebra of $\mathcal{C}(\bar{U}\times\bar{V})$ and $\mathcal{P}_{\bar{U}\times\bar{V}}$ contains the constant fcns. By Stone-Weierstrass, $\mathcal{P}_{\bar{U}\times\bar{V}}$ is dense in $\mathcal{C}(\bar{U}\times\bar{V})$. Let

$$G=\sum_{j=1}^{n} g_j h_j \quad \text{s.t.} \quad \|f-G\|_u < \varepsilon$$

By Urysohn, $\exists \varphi\in\mathcal{C}_c(U)$, $\varphi=1$ on $\pi_x(\operatorname{supp} f)$, $0\le\varphi\le 1$, and $\psi\in\mathcal{C}_c(V)$, $\psi=1$ on $\pi_y(\operatorname{supp} f)$, $0\le\psi\le 1$. Consider

$$F=\varphi\psi G=\sum_{j=1}^{n}(\varphi g_j)(\psi h_j)\in\mathcal{P}.$$

Then, $\operatorname{supp} F\subseteq U\times V$ and $F=G$ on $\pi_x(\operatorname{supp} f)\times\pi_y(\operatorname{supp} f)=K\supseteq\operatorname{supp} f$.

If $(x,y)\in(U\times V)\setminus K$, then $f(x,y)=0$ so

$$|F(x,y)-f(x,y)|=|F(x,y)|\le|G(x,y)|=$$
$$|G(x,y)-f(x,y)|<\|f-G\|_u<\varepsilon. \quad ▨$$

Prop 2. Let μ, ν be Radon measures on X, Y. Then, $\mathcal{C}_c(X\times Y) \subseteq L^1(X\times Y, \mu\times\nu)$ and for $f \in \mathcal{C}_c(X\times Y)$,

$$\int_{X\times Y} f \, d(\mu\times\nu) = \int_Y \Big(\int_X \underbrace{f^y}_{f(\cdot,y)} d\mu \Big) d\nu = \int_X \Big(\int_Y \underbrace{f_x}_{f(x,\cdot)} d\nu \Big) d\mu .$$

Pf. Recall that $\pi_X : X\times Y \to X$ and $\pi_Y : X\times Y \to Y$ are meas. from $\mathcal{B}_X \otimes \mathcal{B}_Y$ to $\mathcal{B}_X$, $\mathcal{B}_Y$, resp. (In fact, this is from the formal def. of $\mathcal{B}_X \otimes \mathcal{B}_Y$.) Also, for $\varphi \in \mathcal{C}_c(X)$, $\psi \in \mathcal{C}_c(Y)$, $(\varphi\psi)(x,y) = \varphi(x)\psi(y) = \big((\varphi\circ\pi_X)(\psi\circ\pi_Y)\big)(x,y)$. $\Rightarrow$ $\varphi\psi$ is $\mathcal{B}_X\otimes\mathcal{B}_Y$ meas. $\Rightarrow$ every $F \in \mathcal{P}$ (from Prop 1) is $\mathcal{B}_X \otimes \mathcal{B}_Y$ meas. Prop 1 $\Rightarrow$ every $f \in \mathcal{C}_c(X\times Y)$ is $\mathcal{B}_X \otimes \mathcal{B}_Y$ meas.

For $f \in C_c(X\times Y)$, $\operatorname{supp} f \subseteq K_X \times K_Y$, where $K_X = \pi_X(\operatorname{supp} f)$, $K_Y = \pi_Y(\operatorname{supp} f)$.

$$\int_{X\times Y} |f| \, d(\mu\times\nu) \le \|f\|_u \, \mu(K_X)\, \nu(K_Y) < \infty$$

$\Rightarrow f \in L^1(\mu\times\nu)$. Since

$$\int_{X\times Y} f \, d(\mu\times\nu) = \int_{K_X\times K_Y} f \, d(\mu\times\nu) = \Big\{ \mu|_{K_X}, \nu|_{K_Y}$$

are finite, in partic. σ-finite; thus, Fubini (Thm. 2.37 (b)) $\Big\} =$

$$\int_{K_Y} \Big(\int_{K_X} f^y \, d\mu \Big) d\nu = \int_{K_X} \Big(\int_{K_Y} f_x \, d\nu \Big) d\mu .$$

The conclusion follows (since $\int_{K_Y} \Big(\int_{K_X} f^y d\mu \Big) d\nu$

$= \int_Y \Big(\int_X f^y \, d\mu \Big) d\nu$ and same for other iterated integral.) $\blacksquare$

Let μ, ν be Radon on X, Y, resp. Unless X, Y are 2^{nd} countable, $\mu \times \nu$ need not be Radon, as already pointed out previously. However, one can construct a unique Radon meas. in its place as follows.

Consider the positive linear functional on $C_c(X \times Y)$: $f \to \int f \, d(\mu \times \nu)$.
(Note that this not a bdd functional, unless $\mu(X) < \infty$, $\nu(Y) < \infty$.)

RRT (basic) $\Rightarrow$ $\exists$ unique Radon meas. on $X \times Y$, called the <u>Radon product</u> of μ and ν and denoted $\mu \hat{\times} \nu$, s.t. $\int f \, d(\mu \times \nu) = \int f \, d(\mu \hat{\times} \nu)$, $f \in C_c(X)$.

II. Fubini-Tonelli Thm for Radon prod.

Recall: • For $E \subseteq X \times Y$, its x-, y-slizes are
$E_x = \{y \in Y : (x,y) \in E\}$, $E^y = \{x \in X : (x,y) \in E\}$.

• As above, $f_x : Y \to \mathbb{C}$ is given by $f_x(y) = f(x,y)$ and $f^y : X \to \mathbb{C}$ by $f^y(x) = f(x,y)$.

We shall need the following basic facts:

Lemma 1 (i) $E \in \mathcal{B}_{X \times Y} \Rightarrow E_x \in \mathcal{B}_Y$, $E^y \in \mathcal{B}_X$.

(ii) $f : X \times Y \to \mathbb{C}$ $\mathcal{B}_{X \times Y}$-meas. $\Rightarrow$
f_x $\mathcal{B}_Y$-meas., f^y $\mathcal{B}_X$-meas.

(iii) $f \in C_c(X \times Y)$, μ, ν Radon on X, Y.
$\Rightarrow$ $x \to \int f_x \, d\nu$, $y \to \int f^y \, d\mu$
are continuous on X, Y.

Pf is DIY, or see Folland Lemmas 7.23 and 7.24.

As in Fubini-Tonelli for product measures, a key ingredient in pf is the following.

<u>Thm2</u>. Let μ, ν be σ-finite Radon meas. on X, Y. (i) If $E \in \mathcal{B}_{X\times Y}$, then $x \to \nu(E_x)$ and $y \to \mu(E^y)$ are $\mathcal{B}_X, \mathcal{B}_Y$-meas. and

$$(\mu \hat{\times} \nu)(E) = \int \nu(E_x)\, d\mu = \int \mu(E^y)\, d\nu.$$

(ii) For $E \in \mathcal{B}_X \otimes \mathcal{B}_Y$, $(\mu \hat{\times} \nu)(E) = (\mu \times \nu)(E)$.

The Fubini-Tonelli Thm for Radon products follows from Thm 2 in the same way FTT for product meas. (Thm 2.37) follows from Thm 2.36. (DIY)

<u>Fubini-Tonelli Thm (Radon products)</u>

Let μ, ν be σ-finite Radon measures on X, Y, and $f \in L^1(X \times Y, \mu \hat{\times} \nu)$.

Then,

(i) $f_x \in L^1(Y, \nu)$ a.e. (μ), $f^y \in L^1(X, \mu)$ a.e. (ν)

(ii) $x \to \int f_x \, d\nu$ is in $L^1(X, \mu)$,

$y \to \int f_y \, d\mu$ is in $L^1(Y, \nu)$.

(iii)

$$\int f \, d(\mu \hat{\times} \nu) = \int \left(\int f_x \, d\nu \right) d\mu = \int \left(\int f^y \, d\mu \right) d\nu.$$

<u>Pf of Thm 2.</u> We shall only do the case when μ, ν are finite. Let $\mathfrak{M}$ be collection of all $E \in \mathcal{B}_{X \times Y}$ for which the conclusion of the Thm (i) holds. Want to show that $\mathcal{B}_{X \times Y} \subseteq \mathfrak{M}$.

To outline the approach, we shall use terminology from Ch. 1-2 in Folland.
Consider $\mathcal{E} = \{A \setminus B : A, B \text{ open in } X \times Y\}$ and let $\mathcal{A} = \{A = \bigcup_{j=1}^{n} A_n : A_i \cap A_j \neq \emptyset, A_j \in \mathcal{E}\}$. We claim $\mathcal{E}$ is an elementary family (see Ch. 1):

- $\emptyset \in \mathcal{E}$
- $E, F \in \mathcal{E} \Rightarrow E \cap F \in \mathcal{E}$
- $E \in \mathcal{E} \Rightarrow E^c = \bigcup_{j=1}^{n} E_j : E_i \cap E_j = \emptyset, E_j \in \mathcal{E}$.

The 1^{st}. is obvious. For 2^{nd}. let A, A', B, B' be open:

$$(A\setminus B)\cap(A'\setminus B') = (A\cap A')\setminus(B\cup B')$$

$\Rightarrow$ $\mathcal{E}$ closed under finite $\cap$.

For 3^{rd}:

$$(A\setminus B)^c = \overset{\in\mathcal{E}}{(X\times Y\setminus A)} \cup \overset{= A\cap B\setminus\emptyset \Rightarrow \in\mathcal{E}}{(A\cap B)}$$

By Prop 1.7, $\mathcal{A}$ is an algebra. By Prop. 2.35, the monotone class generated by $\mathcal{A}$ is also the σ-algebra generated by $\mathcal{A}$. Since $\mathcal{E}$ & $\mathcal{A}$ both are contained in $\mathcal{B}_{X\times Y}$, the σ-alg. gen. by $\mathcal{A}$ is $\mathcal{B}_{X\times Y}$.

The smallest collection of sets $\supseteq \mathcal{A}$ that is closed under increasing unions and decreasing intersections

To show $\mathcal{B}_{X\times Y} \subseteq \mathcal{M}$, it then suffices to show that the monotone class gen. by $\mathcal{A} \subseteq \mathcal{M}$. We shall establish by a series of lemmas.

<u>Lemma 2</u>. $\mathcal{M}$ contains all open sets.

<u>Pf. of Lemma 2</u>. Let $U \subseteq X \times Y$ be open. By Prop 7.11 (from a couple of lectures ago) χ_U is LSC and $\chi_U = \sup\{f : f \prec U\}$ ($f \prec U$ assumes $f \in C_c(X\times Y)$) $\Rightarrow \chi_{U_x} = \sup\{f_x : f \prec U\}$

and $\chi_{U^y} = \sup\{f^y : f \prec U\}$.

By Prop 7.12, ($f_1 \prec U, f_2 \prec U \Rightarrow \max(f_1, f_2) \prec U$)

$$\nu(U_x) = \int \chi_{U_x} d\nu = \sup\left\{\int f_x \, d\nu : f \prec U\right\}$$

$$\mu(U^y) = \sup\left\{\int f^y d\mu : f \prec U\right\}$$

By Lemma 1, $y \to \int f^y d\mu$, $x \to \int f_x d\nu$ are cont., so Prop 7.11 $\Rightarrow$ $\nu(U_x)$, $\mu(U^y)$ are LSC, and in part. $\mathcal{B}_Y$, $\mathcal{B}_X$-meas.

Also, (Prop 7.12 + 7.22)

$$\mu \hat{\times} \nu(U) = \sup\left\{\int f \, d(\mu \hat{\times} \nu) : f \prec U\right\}$$

$$= \{\text{def of } \mu \hat{\times} \nu\} = \sup\left\{\int f \, d(\mu \times \nu) : f \prec U\right\}$$

$$= \sup\left\{\int\left(\int f_x \, d\nu\right) d\mu : f \prec U\right\}$$

$$= \int \sup\left\{\int f_x \, d\nu : f \prec U\right\} d\mu$$

$$= \int \nu(U_x) \, d\mu,$$

and similarly

$$(\mu \hat{\times} \nu)(U) = \int \mu(U^y) \, d\nu$$

$\Rightarrow U \in \mathcal{M}$. $\boxtimes$

Lemma 3. $A, B \in \mathcal{M}$, $B \subseteq A \Rightarrow A \setminus B \in \mathcal{M}$ and $A^c \in \mathcal{M}$.

Pf. Since $A^c = X \setminus A$ and $X \in \mathcal{M}$ by Lemma 2, the 2^{nd} cond. follows from 1^{st}. Also, since we assumed μ, ν finite $\Rightarrow$

- $(\mu \hat{\times} \nu)(A \setminus B) = (\mu \hat{\times} \nu)(A) - (\mu \hat{\times} \nu)(B)$,
- $\mu((A \setminus B)^y) = \mu(A^y \setminus B^y) = \mu(A^y) - \mu(B^y)$,
- $\nu((A \setminus B)_x) = \nu(A_x) - \nu(B_x)$

2^{nd} two $\cdot \Rightarrow x \to \nu((A \setminus B)_x)$, $y \to \mu((A \setminus B)^y)$ are Borel meas. and $A, B \in \mathcal{M}$ + 1^{st} $\cdot \Rightarrow$

$(\mu \hat{\times} \nu)(A \setminus B) = \int \nu(A_x)\, d\mu - \int \nu(B_x)\, d\mu = \int \nu((A \setminus B)^y)\, d\mu$ (and similarly for reverse iterated integral $\Rightarrow A\setminus B \in \mathcal{M}$ as desired.

<u>Rem.</u> • Lemma 3 $\Rightarrow \mathcal{E} \subseteq \mathcal{M}$, for if A, B open, then $A \setminus B = A \setminus (A \cap B)$. Since $A, A\cap B \subseteq A$ open, $A \setminus B \in \mathcal{M}$ by Lemma 3 (and Lemma 2).

• We may also conclude that $\mathcal{A} \subseteq \mathcal{M}$ by noting that, in view of Lemma 3 ($A \in \mathcal{M} \Rightarrow A^c \in \mathcal{M}$), it suffices to check that if $A_1, \dots, A_n \in \mathcal{M}$, $A_i \cap A_j = \emptyset$, then $\bigcup_{j=1}^{n} A_j \in \mathcal{M}$. But this is clear from additivity of measures:

$$(\mu \hat{\times} \nu)\left(\bigcup_1^n A_j\right) = \sum_1^n (\mu \hat{\times} \nu)(A_j).$$

Now, to conclude that the monotone class gen. by $\mathcal{A}$ ($= \mathcal{B}_{X\times Y}$) $\subseteq \mathcal{M}$, it suffices to check that $\mathcal{M}$ is closed under increasing unions: $A_j \in \mathcal{M}$ and

$A_1 \subseteq A_2 \subseteq \dots \subseteq A_n \subseteq \dots$

$\Rightarrow \bigcup_1^\infty A_j \in \mathcal{M}.$

This follows easily from Monotone Conv. Thm,

since $\chi_{A_n} \nearrow \chi_{\bigcup_1^\infty A_n}$ and sim. for slices. This completes the pf of Thm 2 (i).

For $E \in \mathcal{B}_X \otimes \mathcal{B}_Y \subseteq \mathcal{B}_{X \times Y}$, we have

$$(\mu \times \nu)(E) \underset{\text{Tonelli}}{=} \int \left(\int \nu(E_x) \right) d\mu \underset{(i)}{=} \mu \hat{\times} \nu (E).$$

by σ-finiteness

This proves Thm 2 (ii). $\boxtimes$

As mentioned above Fubini-Tonelli for Radon products follows from Thm 2 in the same way FT for products follows from the analog of Thm 2 for products.